

Entropy of Scalar Field near a Schwarzschild Black Hole Horizon

M.R. Setare ^{*}

Department of Science, Payame Noor University. Bijar, Iran

Abstract

In this paper we compute the correction to the entropy of Schwarzschild black hole due to the vacuum polarization effect of massive scalar field. The Schwarzschild black hole is supposed to be confined in spherical shell. The scalar field obeying mixed boundary condition on the spherical shell.

^{*}E-mail: rezakord@ipm.ir

1 Introduction

In general relativity, black hole's properties can be precisely calculated, and the holes may be thought of astronomical objects with masses about several times of sun. In this classical case, event horizon emerges, and anything can not escape from it to arrive at a particular observer who is outside the horizon. But Hawking found that black hole emits radiation and the radiation spectra is just the black body's. This fact indicates that the black hole has a temperature whose expression is [1, 2, 3]

$$T_{BH} = \frac{k}{2\pi}, \quad (1)$$

where k is the surface gravity on the horizon. Thus the Bekenstein-relation becomes the real thermodynamical relationship of the black hole. Namely the black hole has a thermodynamical entropy as

$$S_{BH} = \frac{A}{4l_p^2} \quad (2)$$

where S_{BH} is the Bekenstein-Hawking entropy, A is the area of the event horizon and $l_p = (\frac{\hbar G}{c^3})^{1/2}$ is the Planck length. The presence of quantum field in black hole background modifies the entropy. In the state of thermal equilibrium the total entropy is as [4]

$$S_{tot} = S_{BH} + S_q \quad (3)$$

where S_q is the contribution of radiation and matter fields. Quantum correction to the Bekenstein-Hawking entropy, due to a scalar field, have been computed by different methods for Schwarzschild[5, 6] and Reissner-Nordstrom [7]-[12] black holes.

A very simple but nonetheless instructive model to address the problem of black hole entropy is the so called “brick wall” by ’t Hooft [13]. ’t Hooft considers a quantum field in the background of a classical black hole and using the WKB approximation he derives the thermal entropy of the field outside the horizon of the black hole. In performing the computation, two spatial cutoffs are employed: a large distance one, needed to avoid large volume divergences in the asymptotically flat region, and a short distance one, the “brick wall”, localized just outside the horizon and suppressing the divergences due to the growing number of modes close to the horizon. On the boundaries of the space slice arising in this way, Dirichlet boundary conditions are imposed. The entropy obtained in this model is divergent in the limit of vanishing brick wall thickness. These divergences were later recognized as quantum corrections to the Bekenstein-Hawking formula which can be absorbed into renormalization of the one loop effective gravitational lagrangian [14, 15, 16, 17, 18]

In this letter we would like to investigate the case of Schwarzschild black hole. We consider the massive scalar field obeying mixed boundary condition on a spherical shell, the Schwarzschild black hole is supposed to be confined in the spherical container, in other words we revisit the brick wall model here. Then we obtain the contribution S_q of scalar field onto entropy of a black hole. S_q is due to vacuum polarization effect.

In the generic black hole background the investigation of boundary-induced quantum effects is technically complicated, the exact analytical result can be obtained in the near horizon and large mass limit when the boundary is close to the black hole horizon. In this limit the black hole geometry may be approximated by the Rindler-like manifold (for some

investigations of quantum effects on background of Rindler-like spacetimes see [19, 20, 21]). We consider the vacuum expectation values of the field square and the energy-momentum tensor for a conformally coupled scalar field in the presence of spherical shell on the bulk $Ri \times S^2$, where Ri is a two-dimensional Rindler spacetime.

2 Massive scalar field in Rindler-like spacetime

Let us consider a scalar field $\varphi(x)$ at finite temperature equal to its Hawking value $T = \beta^{-1}$, propagating on background of four-dimensional Rindler-like spacetime $Ri \times S^2$, where Ri is a two-dimensional Rindler spacetime. The corresponding metric is described by the line element

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - r_H^2 d\Sigma_2^2, \quad (4)$$

with the Rindler-like (τ, ξ) part and $d\Sigma_2^2$ is the line element for the space with positive constant curvature with the Ricci scalar $R = 2/r_H^2$. Line element (4) describes the near horizon geometry of four-dimensional AdS black hole with the line element [22, 23]

$$ds^2 = A_H(r) dt^2 - \frac{dr^2}{A_H(r)} - r^2 d\Sigma_2^2, \quad (5)$$

where

$$A_H(r) = k + \frac{r^2}{l^2} - \frac{r_0^3}{l^2 r}, \quad (6)$$

and $k = 0, -1, 1$. In (6) the parameter l is related to the bulk cosmological constant and the parameter r_0 depends on the mass of the black hole and on the bulk gravitational constant. In the non extremal case the function $A_H(r)$ has a simple zero at $r = r_H$. In the near horizon limit, introducing new coordinates τ and ρ in accordance with

$$\tau = \frac{1}{2} A'_H(r_H) t, \quad r - r_H = \frac{1}{4} A'_H(r_H) \xi^2, \quad (7)$$

the line element is written in the form (4). Note that for a (3+1)-dimensional Schwarzschild black hole one has $A_H(r) = 1 - \frac{r_H}{r}$ and, hence, $A'_H(r_H) = 1/r_H$. The field equation is in the form

$$(g^{ik} \nabla_i \nabla_k + m^2 + \zeta R) \varphi(x) = 0, \quad (8)$$

where ζ is the curvature coupling parameter. Below we will assume that the field satisfies the Robin boundary condition

$$\left(A + B \frac{\partial}{\partial \xi} \right) \varphi = 0, \quad \xi = a, \quad (9)$$

on the hypersurface $\xi = a$, with constant coefficients A and B . In accordance with (7) this hypersurface corresponds to the spherical shell near the black hole horizon with the radius $r_a = r_H + A'_H(r_H)a^2/4$.

A black hole behaves like a thermodynamic system and possesses temperature and entropy. As we have mentioned in introduction in the state of thermal equilibrium the total entropy is as Eq.(3). The Euclidean action for scalar filed in our interesting background takes the standard thermodynamic form

$$I_q = -\beta \int d^3x \sqrt{g} \langle 0_0 | T_0^0 | 0_0 \rangle - S_q \quad (10)$$

where T_μ^ν is the stress tensor of the quantum field calculated with respect to the background (5), g is determinant of the metric. From the conservation law $T_{\nu;\mu}^\mu = 0$ in the background (5) it follows that [24]

$$\frac{r}{2} \left(< T_1^1 > - < T_0^0 > \right) A'_H = \frac{1}{r_H} [(r^3 < T_1^1 >)_1 - r^2 < T_i^i >] \quad (11)$$

where i denotes spatial indices. Then one can obtain (for more details see [24])

$$(4\pi)^{-2} A'_H(r_H) \frac{\partial I_q}{\partial r_H} = \int_{r_H}^{r_a} dr r^2 < T_i^i > - r_a^3 < T_1^1 > + r_H^3 [< T_1^1(r_H) > - < T_0^0(r_H) >] \quad (12)$$

The regularity of $< T_\mu^\nu >$ at the event horizon in the Hartle-Hawking state demands that according to (11) $< T_1^1(r_H) > = < T_0^0(r_H) >$, so the last term in (12) is square brackets cancels. The above formula are applicable to any field with a tensor $< T_\mu^\nu >$. Now we consider the Schwarzschild background case, although one can consider the general line element (5), the Schwarzschild is only a simple case, in this case we have

$$(4\pi)^{-2} \frac{\partial I_q}{\partial r_H} = r_a^3 < T_r^r > - \int_{r_H}^{r_a} dr r^2 < T_i^i > \quad (13)$$

3 Vacuum expectation values of energy-momentum tensor and entropy of scalar field

Now we consider the vacuum expectation values (VEV) of the energy-momentum tensor for the geometry $R^2 \times S^2$ described by the following line element

$$ds^2 = dt^2 - (dx^1)^2 - r_H^2 d\Sigma_2^2, \quad (14)$$

where the coordinates (t, x^1) are related to the coordinates (τ, ξ) by $t = \xi \sinh \tau$, $x^1 = \xi \cosh \tau$. This line element describes the near horizon geometry of a non-extremal black hole spacetime. We will consider the vacuum expectation values for the geometry without boundaries. The corresponding Wightman function can be presented in the form [25]

$$\begin{aligned} G_0^+(x, x') &= \tilde{G}_0^+(x, x') - \frac{\Gamma(3/2)}{2\pi^{7/2} r_H^2} \sum_{l=0}^{\infty} (2l+1) C_l^{1/2}(\cos \theta) \\ &\quad \times \int_0^{\infty} d\omega e^{-\omega \pi} \cos[\omega(\tau - \tau')] K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi'), \end{aligned} \quad (15)$$

with the function

$$\begin{aligned} \tilde{G}_0^+(x, x') &= \frac{\Gamma(3/2)}{2\pi^{7/2} r_H^2} \sum_{l=0}^{\infty} (2l+1) C_l^{1/2}(\cos \theta) \\ &\quad \times \int_0^{\infty} d\omega \cosh\{\omega[\pi - i(\tau - \tau')]\} K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi'). \end{aligned} \quad (16)$$

where $K_{i\omega}(\lambda_l \xi)$, is the Bessel modified function with the imaginary order, and $C_l^{1/2}(\cos \theta)$ is the Gegenbauer or ultraspherical polynomial of degree l and order $1/2$. In (15), the divergences in the coincidence limit are contained in the term $\tilde{G}_0^+(x, x')$, the amplitude of

the corresponding vacuum state we will denote by $|\tilde{0}\rangle$. First of all we note that from the problem symmetry it follows that the expectation values $\langle \tilde{0} | T_i^i | \tilde{0} \rangle$ do not depend on the point of observation and

$$\begin{aligned}\langle \tilde{0} | T_0^0 | \tilde{0} \rangle &= \langle \tilde{0} | T_1^1 | \tilde{0} \rangle, \\ \langle \tilde{0} | T_2^2 | \tilde{0} \rangle &= \langle \tilde{0} | T_3^3 | \tilde{0} \rangle.\end{aligned}\quad (17)$$

The component $\langle \tilde{0} | T_2^2 | \tilde{0} \rangle$ can be expressed through the energy density by using the trace relation

$$T_i^i = 3(\zeta - \zeta_c) \nabla_i \nabla^i \varphi^2 + m^2 \varphi^2. \quad (18)$$

From this relation it follows that

$$\langle \tilde{0} | T_2^2 | \tilde{0} \rangle = \frac{1}{2} [m^2 \langle \tilde{0} | \varphi^2 | \tilde{0} \rangle - 2 \langle \tilde{0} | T_0^0 | \tilde{0} \rangle]. \quad (19)$$

Hence, it is sufficient to find the renormalized vacuum expectation values of the field square and the energy density. Then as have shown in [25]

$$\langle \tilde{0} | \varphi^2 | \tilde{0} \rangle = \frac{\zeta(1/2)\Gamma(3/2)}{8\pi^{5/2} r_H^2}, \quad \langle \tilde{0} | T_0^0 | \tilde{0} \rangle = \frac{\zeta(-1/2)\Gamma(3/2)}{8\pi^{5/2} r_H^4} \quad (20)$$

Using Eqs.(17,19) we obtain

$$\langle \tilde{0} | T_i^i | \tilde{0} \rangle = 2 \langle \tilde{0} | T_0^0 | \tilde{0} \rangle + 2 \langle \tilde{0} | T_2^2 | \tilde{0} \rangle = m^2 \langle \tilde{0} | \varphi^2 | \tilde{0} \rangle \quad (21)$$

Now we turn to the VEV of the energy-momentum tensor. The corresponding operator we will take in the form

$$T_{ik} = \partial_i \varphi \partial_k \varphi + \left[\left(\zeta - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \zeta \nabla_i \nabla_k - \zeta R_{ik} \right] \varphi^2, \quad (22)$$

with the trace relation (18). In (22) R_{ik} is the Ricci tensor for the bulk geometry and for the metric (4) it has components

$$R_{ik} = 0, \quad i, k = 0, 1; \quad (23)$$

$$R_{ik} = \frac{n}{r_H^2} g_{ik}, \quad i, k = 2, 3. \quad (24)$$

On the base of formula (22) the corresponding vacuum expectation values are presented in the form

$$\langle 0_0 | T_{ik} | 0_0 \rangle = \lim_{x' \rightarrow x} \nabla_i \nabla'_k G_0^+(x, x') + \left[\left(\zeta - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \zeta \nabla_i \nabla_k - \zeta R_{ik} \right] \langle 0_0 | \varphi^2 | 0_0 \rangle. \quad (25)$$

By using decomposition (15), the vacuum energy-momentum tensor is presented in the form

$$\langle 0_0 | T_{ik} | 0_0 \rangle = \langle \tilde{0} | T_{ik} | \tilde{0} \rangle + \langle T_{ik} \rangle^{(0)}, \quad (26)$$

where the second summand on the right is given by formula [25]

$$\langle T_i^k(x) \rangle^{(0)} = \frac{-\delta_i^k \Gamma(3/2)}{2\pi^{7/2} r_H^2} \sum_{l=0}^{\infty} D_l \lambda_l^2 \int_0^\infty d\omega e^{-\omega\pi} f^{(i)} [K_{i\omega}(\lambda_l \xi)]. \quad (27)$$

In this formula we use the notations

$$\begin{aligned} f^{(0)}[g(z)] &= \left(\frac{1}{2} - 2\zeta\right) \left[\left(\frac{dg(z)}{dz}\right)^2 + \left(1 - \frac{\omega^2}{z^2}\right) g^2(z) \right] + \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \frac{\omega^2}{z^2} g^2(z), \quad (28) \\ f^{(1)}[g(z)] &= -\frac{1}{2} \left(\frac{dg(z)}{dz}\right)^2 - \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \frac{1}{2} \left(1 - \frac{\omega^2}{z^2}\right) g^2(z), \\ f^{(i)}[g(z)] &= \left(\frac{1}{2} - 2\zeta\right) \left[\left(\frac{dg(z)}{dz}\right)^2 + \left(1 - \frac{\omega^2}{z^2}\right) g^2(z) \right] - \frac{\lambda_l^2 - m^2}{2\lambda_l^2} g^2(z), \end{aligned}$$

also we use the following notations

$$D_l = \frac{(2l+1)\Gamma(l+1)}{l!} \quad (29)$$

$$\lambda_l = \sqrt{\frac{l(l+1) + 2\zeta}{r_H^2} + m^2}. \quad (30)$$

Now by considering Eq.(13)we can write

$$(4\pi)^{-2} \frac{\partial I_q}{\partial r_H} = r_a^3 \langle 0_0 | T_1^1 | 0_0 \rangle - \int_{r_H}^{r_a} dr r^2 \langle 0_0 | T_i^i | 0_0 \rangle \quad (31)$$

then using Eqs.(20,21,26) we obtain

$$\begin{aligned} I_q &= \frac{\Gamma(3/2)}{6\pi^{1/2}} [m^2 \zeta(1/2) (\frac{r_a^3}{r_H} + 1/2r_H^2) - \frac{\zeta(-1/2)}{r_H^3}] + \frac{r_a^3}{16\pi^2} \int dr_H \langle T_1^1 \rangle^{(0)} \\ &- \frac{1}{16\pi^2} \int dr_H \int_{r_H}^{r_a} dr r^2 \langle T_i^i \rangle^{(0)} + c \end{aligned} \quad (32)$$

where c is a constant. For small values ξ the vacuum expectation values (27) behave as $(\frac{r_H}{\xi})^4$, therefore

$$I_q = \frac{\Gamma(3/2)}{6\pi^{1/2}} [m^2 \zeta(1/2) (\frac{r_a^3}{r_H} + 1/2r_H^2) - \frac{\zeta(-1/2)}{r_H^3}] + \frac{r_H^5}{48\pi^2 \xi^4} (r_a^3/5 - r_H^3/2) + c \quad (33)$$

Now using Eq.(10) we have

$$\begin{aligned} S_q &= \frac{-\beta(r_a^3 - r_H^3)}{6\pi^{3/2} r_H^4} \zeta(-1/2) \Gamma(3/2) + \frac{4\pi\beta(r_a^3 - r_H^3)}{3} \frac{r_H^4}{\xi^4} \\ &- \frac{\Gamma(3/2)}{6\pi^{1/2}} [m^2 \zeta(1/2) (\frac{r_a^3}{r_H} + 1/2r_H^2) - \frac{\zeta(-1/2)}{r_H^3}] - \frac{r_H^5}{48\pi^2 \xi^4} (r_a^3/5 - r_H^3/2) + c \end{aligned} \quad (34)$$

We have succeeded in obtaining a reasonable expression, valid only for very large black hole mass and near the horizon. On the horizon the vacuum expectation values (26,27) diverge. These surface divergences are well known in quantum field theory with boundaries and are investigated for various type boundary conditions and geometries. The leading term in the near horizon asymptotic expansions behaves as $(\frac{r_H}{\xi})^4$ for the components of the energy-momentum tensor. On the horizon $\xi = 0$ and $r_a = r_H$, then one can see that the divergent term in the entropy S_q is proportional to the $\frac{1}{\xi^4}$. A possible way to deal

with such divergences has been suggested in [13], where it has been argued that the quantum fluctuations at the horizon might provide a natural cutoff. In Ref.[13], 't Hooft attempted to provide a microphysical explanation of black hole entropy by considering the modes for a scalar field in the vicinity of a black hole. In such a calculation, one finds a divergence in the number of modes because of the infinite blue shift at the event horizon. To regulate his calculation, 't Hooft introduced a “brick wall” cut-off, demanding that the scalar field vanish within some fixed distance of the horizon. 't Hooft introduced this “simple-minded” cut-off as an attempt to mimic what he hoped would be a true physical regulator arising from gravitational interactions. Susskind and Uglum suggested that the entropy divergences have the correct form to be absorbed in the Bekenstein-Hawking formula as a renormalization of Newton's constant[14]. Thus these calculations should be regarded as yielding the one-loop correction of quantum field theory to the black hole entropy[26, 27, 28].

4 Conclusion

In this paper we studied the quantum vacuum effects to the entropy produced by a spherical shell in 4–dimensional $Ri \times S^2$ spacetime, with Ri being a two-dimensional Rindler spacetime. The corresponding line element (4) describes the near horizon geometry of a non-extremal black hole spacetime defined by the line element (5). The case of a massive scalar field with conformal coupling parameter and satisfying the Robin boundary condition on the sphere is considered. Then by considering the energy-momentum tensor of quantum massive scalar field near the horizon of Schwarzschild black hole and using general formula (12), we recovered the contribution S_q of these field into the entropy of a black hole. The vacuum expectation values of the field square and the energy-momentum tensor are expressed in terms of the zeta function by formulas (20), at $s = 1/2$ and $s = -1/2$ the zeta function $\zeta(s)$ has simple poles with residues $\zeta_{S^2}(0)$ and $\frac{\zeta_{S^2}(-1)}{2}$, respectively. Hence, in general, the vacuum expectation values of the field square and the energy density contain the pole and finite contributions. The remained pole term is a characteristic feature for the zeta function regularization method. As a result the vacuum expectation values of the energy-momentum tensor for the boundary-free geometry are determined by formulas (26), (27). On the horizon these expectation values diverge. The leading terms in the near horizon asymptotic expansion behave as $(\frac{r_H}{\xi})^4$ for the components of the energy-momentum tensor. On the horizon $\xi = 0$ and $r_a = r_H$, then one can see that the divergent term in the entropy S_q is proportional to the $\frac{1}{\xi^4}$. These surface divergences are well known in quantum field theory with boundaries and are investigated for various type boundary conditions and geometries. Local surface divergences were first discussed for arbitrary smooth boundaries by Deutsch and Candelas [29]. They found cubic divergences in the energy density as one approaches the surface; for example, outside a Dirichlet sphere (that is, for a conformally-coupled scalar field satisfying Dirichlet boundary conditions on the surface) the energy density diverges. The first calculations on the problem of divergences in one-loop thermodynamical quantities for matter fields in thermal equilibrium on a black hole background are due to G. 't Hooft [13]: using a WKB approximation for the eigenvalues of a scalar field hamiltonian on the Schwarzschild background, 't Hooft finds that thermodynamical quantities as free energy, internal energy and entropy have contributions divergent for the radial coordinate $r \rightarrow r_H$. So

one must introduce a short-distance cut-off ϵ representing a radial proper distance from the horizon. The divergences in the thermodynamical quantities behave as ϵ^{-2} . ‘t Hooft proposal to face with these divergences is the so called brick wall model.

References

- [1] S. W. Hawking, Nature, **248**, 30, (1974).
- [2] S. W. Hawking, Commun. Math. Phys. **43**, 199, (1975).
- [3] S. W. Hawking, Phys. Rev. **D14**, 2460, (1976).
- [4] O. B. Zaslavskii, gr-qc/0101055.
- [5] G. Cognola, L. Vanzo and S. Zerbini, Class. Quant. Grav. **12**, 1927, (1995).
- [6] V. P. Frolov, D. V. Furasev and A. I. Zelnikov, Phys. Rev. **D54**, 2711, (1996).
- [7] J. Demers, R. Lafrance and R. C. Myers, Phys. Rev. **D52**, 2245, (1995).
- [8] A. Ghosh and P. Mitra, Phys. Lett. **B357**, 295, (1995).
- [9] G. Cognola, L. Vanzo and S. Zerbini, Phys. Rev. **D52**, 4548, (1995).
- [10] A. Ghosh and P. Mitra, Mod. Phys. Lett. **A11**, 2933, (1996).
- [11] S. P. de Alwis and N. Ohta, hep-th/9504033.
- [12] A. Romeo, Class. Quant. Grav. **13**, 2797, (1996).
- [13] G. ’t Hooft, Nucl. Phys. **B256**, 727 (1985).
- [14] L. Susskind and J. Uglum, Phys. Rev. **D50**, 2700, (1994).
- [15] D. V. Fursaev and S. N. Solodukhin, Phys. Lett. **B365**, 51, (1996).
- [16] E. Winstanley, Phys. Rev. **D 63**, 084013, (2001).
- [17] R. Brustein, M. B. Einhorn, and A. Yarom, hep-th/0508217.
- [18] G. Milanesi, M. Mintchev, hep-th/0509080.
- [19] A. A. Bytsenko, G. Cognola, and S. Zerbini, Nucl. Phys. **B458**, 267, (1996).
- [20] S. Zerbini, G. Cognola, and L. Vanzo, Phys. Rev. **D54**, 2699, (1995).
- [21] G. Cognola, E. Elizalde and S. Zerbini, Phys. Lett. **B585**, 155, (2004).
- [22] R. B. Mann, Class. Quant. Grav. **16**, L109, (1997).
- [23] L. Vanzo, Phys. Rev. **D56**, 6475, (1997).
- [24] O. B. Zaslavskii, Class. Quant. Grav. **13**, L23, (1996).

- [25] A. A. Saharian, M. R. Setare, Nucl. Phys. **B724**, 406, (2005).
- [26] C. Callan and F. Wilczek, *Phys. Lett.* **B333**, 55, (1994).
- [27] T. Jacobson, *Phys. Rev.* **D50**, 6031, (1994).
- [28] J. G. Demers, R. Lafrance, R. C. Myers, *Phys. Rev.* **D52**, 2245, (1995).
- [29] D. Deutsch and P. Candelas , *Phys. Rev.* **D20**, 3063, (1979).